A Cyclic Reduction Method for Solving Algebraic Riccati Equations

D. A. Bini and B. Iannazzo

December 23, 2006

Abstract

A new iterative method for solving algebraic Riccati equations is presented. The algorithm is based on the transformation of the Riccati equation into an equation of the form $AX^2 + BX + C = 0$ which is efficiently solved via the Cayley transform and cyclic reduction. The algorithm is quadratically convergent and it is faster than the currently available methods.

1 Introduction

Let $D, A, C \in \mathbb{R}^{n\times n}$ be $n \times n$ matrices such that $D^T = D, C^T = C$ and consider the continuous-time Algebraic Riccati Equation (ARE)

$$XDX + A^T X + XA - C = 0, \tag{1}$$

where the unknown X is an $n \times n$ matrix. We call D the *leading coefficient* of (1). With a dualism inherited from control theory in which Riccati equation is encountered, we can define a Discrete-time Algebraic Riccati Equation (DARE) as

$$A^T XA - X + Q - (C + B^T XA)^T (R + B^T XB)^{-1}(C + B^T XA) = 0, \tag{2}$$

where A, Q, $X \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{m\times n}$ with $m \le n$ and $R = R^T$, $Q = Q^T$; we call B the *leading coefficient* of (2).

Both equations (1) and (2) are very important for their application to optimal control [17].

The main computational problem, motivated by the applications, is to search for a solution X of (1) which is stabilizing, i.e., $\sigma(A + DX) \subset \mathbb{C}^-$, where $\sigma(B)$ is the set of the eigenvalues of B and $\mathbb{C}^-$ is the set of complex numbers with negative real part. For the equation (2), we search for a solution X with the property $\sigma(A - B(R + B^T XB)^{-1}(C + B^T XA)) \subset \mathcal{D}^-$, where $\mathcal{D}^-$ is the set of complex numbers with modulus less than 1.

Another matrix equation of great importance is the Quadratic Matrix Equation (QME)

$$AX^2 + BX + C = 0, \quad A, B, C, X \in \mathbb{R}^{n\times n}. \tag{3}$$

which plays a crucial role in the numerical solution of certain Markov chains [3, 16], where the interest is focused on the *minimal* solutions Y, i.e., such that their spectral radius $\rho(Y)$ is minimal among all the other solutions.

Regarding Riccati equations there is a wide literature with many algorithms, a great number of articles and some books [4, 13, 17]. On the other hand, the attention devoted to QMEs has been comparably minor. Among the papers on QME it is important to cite the nice review of [10] and references given therein. It is interesting to point out that some algorithms for QME are derived from algorithms for ARE. Despite the minor interest addressed to QMEs, very efficient algorithms for solving QMEs have been recently designed [3, 15] in the context of Markov chains.

In this paper, we move in the opposite way, that is, we try to *translate* algorithms developed for QME into algorithms for Riccati equations, in particular we focus our attention onto Cyclic Reduction (CR) which, for the computation of the minimal solutions of QME, represents the state of the art among the algorithms developed until now [3].

Under the assumption of nonsingularity of the leading coefficient, we determine suitable transformations from equations (1) and (2) to equation (3) which allow one to derive new algorithms for ARE and DARE directly from algorithms for QME. The nonsingularity assumption of the leading coefficient is successively relaxed simply by considering a new Riccati equation with slightly larger coefficients having a nonsingular leading coefficient. This equation is constructed so that we may easily derive the solution of the original equation from the solution of the latter.

We show that computing a stabilizing solution for (1) and (2), can be reduced to computing a stabilizing solution of a suitable QME. Moreover, by using the Cayley transform, we reduce the latter computation to computing the minimal solution of a different QME. For the latter computation we propose an adaptation of the cyclic reduction algorithm of [3].

The convergence of the algorithm obtained in this way is generally quadratic and still holds in certain limit cases, say, when the Hamiltonian matrix has pure imaginary eigenvalues. The numerical experiments, performed with a wide set of test problems, show that our algorithm is faster than the other methods currently available. For certain problems, the speed up factor of the cpu time is greater than 10.

Comparisons have been performed with the main methods currently available for Riccati equations, i.e., Schur's method [12], the Matrix Sign method [5, 18] and Newton's iteration [6, 11], based on the benchmark of [1].

The paper is organized as follows. In section 2 we show how an algebraic Riccati equation can be transformed into a suitable QME and recall the cyclic reduction method. In section 3 we provide our algorithm for ARE and DARE, respectively, in the case where the leading coefficient is nonsingular. In section 4 we provide the algorithm for the general case and in section 5 we show its efficiency by means of some numerical experiments.

2 Reduction to a QME and Cyclic Reduction

Assume D nonsingular and let X be a solution of the continuous-time algebraic Riccati equation (1). It can be easily proved by direct inspection that $Z = A + DX$ solves the QME

$$D^{-1}Z^2 + (A^T D^{-1} - D^{-1}A)Z - C - A^T D^{-1}A = 0, \tag{4}$$

and conversely, if Z solves equation (4) then $X = D^{-1}(Z - A)$ is a solution of (1). Moreover, the stabilizing condition $\sigma(A + DX) \subset \mathbb{C}^-$ for X turns into $\sigma(Z) \subset \mathbb{C}^-$. Therefore, in order to find a stabilizing solution X of (1), it can be computed a solution Z of (4) with $\sigma(Z) \subset \mathbb{C}^-$ which yields $X = D^{-1}(Z - A)$.

Similar transformations hold for nonsymmetric algebraic Riccati equations but in this paper we will focus only on symmetric ones.

Cyclic Reduction (CR) is an algorithm developed in the late sixties by G. Golub [8], for the solution of certain block tridiagonal block Toeplitz systems encountered in the numerical treatment of Poisson's equations [7]. More recently, CR has been used for the effective solution of QMEs [3].

Let us recall briefly how cyclic reduction works for the solution of the QME (3), we refer the reader to the papers [2, 3] for more details.

Starting with $A_0 = A$, $B_0 = B$, $C_0 = C$, $\widetilde{B}_0 = B$ and $\widehat{B}_0 = B$, cyclic reduction generates the sequences $\{A_i\}_{i\in\mathbb{N}}$, $\{B_i\}_{i\in\mathbb{N}}$, $\{C_i\}_{i\in\mathbb{N}}$ $\widetilde{B}_i$ and $\{\widehat{B}_i\}_{i\in\mathbb{N}}$, by means of the following relations

$$\begin{cases} B_{i+1} = B_i - A_i B_i^{-1} C_i - C_i B_i^{-1} A_i, \\ A_{i+1} = -A_i B_i^{-1} A_i, \quad C_{i+1} = -C_i B_i^{-1} C_i, \\ \widehat{B}_{i+1} = \widehat{B}_i - A_i B_i^{-1} C_i, \quad \widetilde{B}_{i+1} = \widetilde{B}_i - C_i B_i^{-1} A_i. \end{cases} \tag{5}$$

Here and hereafter we assume that the matrices B_i are invertible for $i \in \mathbb{N}$.

It is known [9] that solutions of equation (3) are related to the generalized eigenvalues of the matrix polynomial $P(\lambda) = A\lambda^2 + B\lambda + C$, that is, the zeros of the polynomial $\det P(\lambda)$. In particular, each solution of (3) has eigenvalues which are zeros of $\det P(\lambda)$ and equation (3) has a solution X with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if the Jordan chains of the matrix polynomial $P(\lambda)$ related to the eigenvalues $\lambda_1, \ldots, \lambda_n$ are a set of linearly independent vectors.

The following result of [2] relates the asymptotic behavior of the matrix sequences generated by cyclic reduction with the solution of the QME (3) with the minimal spectral radius.

Theorem 1. *Let A_i, B_i, C_i, $\widehat{B}_i$, $\widetilde{B}_i$, $i \geq 0$, be the matrices generated by the CR algorithm applied to the equation (3), where $\det(B_i) \neq 0$ so that CR can be carried out. Assume that the quadratic matrix equations $AX^2 + BX + C = 0$ and $CY^2 + BY + A = 0$ have the two minimal solutions X and Y with spectral radius $\rho(X) < 1$, $\rho(Y) < 1$. Then for any matrix norm $\|\cdot\|$, the sequences $\|B_i\|$, $\|B_i^{-1}\|$, $\|\widehat{B}_i^{-1}\|$ and $\|\widetilde{B}_i^{-1}\|$ are bounded from above by a constant and for any $\sigma < 1$ such that $\rho(X) < \sigma$ and $\rho(Y) < \sigma$, it holds $\|A_i\| = O(\sigma^{2^j})$ and $\|C_i\| = O(\sigma^{2^j})$. Moreover, $X = -\lim_i \widehat{B}_i^{-1}C$, $Y = -\lim_i \widetilde{B}_i^{-1}A$, and $\|X + \widehat{B}_i^{-1}C\| = O(\sigma^{2^i})$, $\|X + \widetilde{B}_i^{-1}A\| = O(\sigma^{2^i})$.*

The convergence properties of cyclic reduction can be better expressed in terms of the generalized eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ of the matrix polynomial $P(\lambda)$ where we count zeros at infinity if the leading matrix A is singular. Assume that these zeros are ordered by modulus so that $|\lambda_1| \le \cdots \le |\lambda_n| < 1 < |\lambda_{n+1}| \le \cdots \le |\lambda_{2n}|$, Then $\rho(X) = |\lambda_n|$, $\rho(Y) = 1/|\lambda_{n+1}|$ and the constant γ in the above theorem is such that $|\lambda_n/\lambda_{n+1}| < \gamma < 1$. So that the farther are the generalized eigenvalues from the unit circle, the lower is the number of steps required to obtain an approximate solution with an assigned precision.

CR provides approximations to the solution with eigenvalues in the unit disk. But in our problem we need to find solutions X of (3) with eigenvalues in the left complex half-plain $\mathbb{C}^-$. To overcome this problem we use the Cayley transform $z \to (z+1)/(z-1)$ which maps the left open half plane $\mathbb{C}^-$ in the open unit disk, the imaginary axis in the unit circle, and the right half plane outside the unit disk. This transform can be applied also to matrix equations, and its effects are synthesized by the following simple result of which we omit the proof.

Theorem 2. *Let $A_1, A_0, A_{-1}, Z \in \mathbb{R}^{n\times n}$ with $Z-I$ invertible and set $T = (Z+I)(Z-I)^{-1}$. Then Z is a solution of $A_1Z^2 + A_0Z + A_{-1} = 0$, with eigenvalues having negative real part if and only if T is solution of $B_1T^2 + B_0T + B_{-1} = 0$ with spectral radius less than one, where $B_1 = A_1 + A_0 + A_{-1}$, $B_0 = 2(A_1 - A_{-1})$, $B_{-1} = A_1 - A_0 + A_{-1}$. In particular λ is an eigenvalue of Z if and only if $\mu = \frac{\lambda+1}{\lambda-1}$ is an eigenvalue of T.*

The way to find a solution with eigenvalues in the left half plane is to transform the equation and then find the corresponding solution with spectral radius less than one.

If cyclic reduction can be applied to the transformed equation then convergence is still quadratic. The following result proves that cyclic reduction can be successfully applied.

Theorem 3. *Assume that the matrix equations $A_1X^2 + A_0X + A_{-1} = 0$ and $A_1Y^2 - A_0Y + A_{-1} = 0$ have solutions X and Y with eigenvalues in the left half plane of $\mathbb{C}$. Then the generalized eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ of $P(\lambda) = A_1\lambda^2 + A_0\lambda + A_{-1}$ can be ordered so that*

$$\mu_1 \le \cdots \le \mu_n < 1 < \mu_{n+1} \le \cdots \le \mu_{2n} \tag{6}$$

for $\mu_i = |\lambda_i + 1|/|\lambda_i - 1|$, where some of the rightmost terms can be infinity if for some i, it holds $\lambda_i = 1$. Moreover, if CR can be carried out, when applied to the matrix equation $B_1T^2 + B_0T + B_{-1} = 0$ obtained from $A_1X^2 + A_0X + A_{-1} = 0$ by means of the Cayley transform, then $X = X_i + O(\tau^{2^i})$ for any τ such that

$$\mu_n/\mu_{n+1} < \tau < 1, \tag{7}$$

and $X_i = (T_i + I)(T_i - I)$ where $T_i = -\widehat{B}_i^{-1}B_{-1}$ and $\widehat{B}_i$ is obtained by means of CR as in (5).

Proof. Applying the Cayley transform to the matrix equations $A_1X^2 + A_0X + A_{-1} = 0$ and $A_1Y^2 - A_0Y + A_{-1} = 0$ and to their solution X and Y, yields the new equations $B_1T^2 + B_0T + B_{-1} = 0$ and $B_{-1}W^2 + B_0W + B_1 = 0$ which have solutions T and W, respectively, with spectral radius less than one (compare with theorem 2). So from theorem 1 it follows that CR converges. From theorem 2, the eigenvalues of the matrix polynomial $B_1\xi^2+B_0\xi+B_{-1}$ are the numbers $\xi_i = (\lambda_i + 1)/(\lambda_i - 1)$, and from the existence of solutions T and W it follows that half of the ξ_i are inside the unit circle and half outside. So they can be ordered like (6) and $T = T_i + O(\tau^{2^i})$, $W = W_i + O(\tau^{2^i})$ where T_i and W_i are obtained via cyclic reduction. It remains to prove that $X = X_i + O(\tau^{2^i})$. Let us choose a suitable norm, so there exist constants H, K and an integer $i_0 > 0$ such that $\|T - T_i\| \leq K\tau^{2^i},\ \|(T - I)^{-1}\| < H,\ \|(T_i - I)^{-1}\| < H,\ i > i_0$, in fact T has no eigenvalues on the unit circle and T_i tends to T. Now, since $X - X_i = (T - I)^{-1}(T + I) - (T_i + I)(T_i - I)^{-1} = (T - I)^{-1}((T + I)(T_i - I) - (T - I)(T_i + I))(T_i - I)^{-1}$, then $\|X - X_i\| \leq 2\|(T - I)^{-1}\| \cdot \|(T_i - I)^{-1}\| \cdot \|T - T_i\| \leq \alpha\|T - T_i\| \leq \alpha K\tau^{2^i}$, with α being a constant. This completes the proof.

Let us describe the case of symmetric equations that are of interest for our study. We say that a quadratic matrix equation (3) is symmetric in the domain of the imaginary axis or *continuous-symmetric* if A and C are symmetric matrices and B is a skew-symmetric matrix ($A^T = A$, $B^T = -B$, $C^T = C$), we say that the equation is symmetric in the domain of the unitary circle or *discrete-symmetric* if $A^T = C$ and $B^T = B$.

For the matrix polynomial $P(\lambda) = A\lambda^2 + B\lambda + C$, associated with the above matrix equation, these symmetries turn into $P(-\lambda) = P(\lambda)^T$ and $P(1/\lambda)\lambda^2 = P(\lambda)^T$, respectively.

It can be easily verified that applying the Cayley transform to a continuous-symmetric, quadratic matrix equation yields a quadratic matrix equation which is discrete-symmetric.

It is important to point out that if $C_0 = A_0^T$ and $B_0^T = B_0$, then for the sequences of matrices generated by cyclic reduction it holds $C_i = A_i^T$ and $B_i^T = B_i$, $i = 1, 2, \ldots$, so that cyclic reduction is less expensive.

3 Algorithms

Let us consider the problem of finding a solution X of the continuous-time algebraic Riccati equation $XDX + A^TX + XA - C = 0$, such that $\sigma(A + DX) \subset \mathbb{C}^-$. With the replacement $Z = A + DX$ and with the hypotheses that the leading coefficient D is invertible, we need to find a solution Z, such that $\sigma(Z) \subset \mathbb{C}^-$, of the continuous-symmetric QME

$$D^{-1}Z^2 + (A^TD^{-1} - D^{-1}A)Z - C - A^TD^{-1}A. \tag{8}$$

Applying the Cayley transform $T = (Z + I)(Z - I)^{-1}$ and theorem 2 we obtain the new equation that is discrete-symmetric $KT^2 + HT + K^T = 0$, where $K =$

$(I + A^T)D^{-1}(I - A) - C$, $H = 2(D^{-1} + C + A^T D^{-1} A)$. The task is to find a solution T such that $\sigma(T) \subset \mathcal{D}^-$, and this can be performed through symmetric cyclic reduction. The final algorithm is described below

Algorithm 1. *Cyclic reduction for continuous-time algebraic Riccati equations*

1. Input: D, A, C with $D^T = D, C^T = C, D$ invertible.

2. Put $H_0 = 2(D^{-1}+C+A^T D^{-1}A)$, $\widehat{H}_0 = \widetilde{H}_0 = H_0$, $K_0 = (I+A^T)D^{-1}(I-A) - C$.

3. Repeat the symmetric cyclic reduction step
$H_{i+1} = H_i - K_i H_i^{-1} K_i^T - K_i^T H_i^{-1} K_i$
$K_{i+1} = -K_i H_i^{-1} K_i, \widehat{H}_{i+1} = \widehat{H}_i - K_i H_i^{-1} K_i^T$, $\widetilde{H}_{i+1} = \widetilde{H}_i - K_i^T H_i^{-1} K_i$
until $\|K_i\| < \varepsilon$, with a given ε and a given matrix norm.

4. Compute $T_i = -\widehat{H}_i^{-1} K$, $W_i = -\widetilde{H}_i^{-1} K^T$, $X_i = D^{-1}((T_i + I)(T_i - I)^{-1} - A)$, and $Y_i = -D^{-1}((W_i + I)(W_i - I)^{-1} + A)$.

5. Output: approximated solutions of the Riccati equation such that $\sigma(A + DX_i) \subset \mathbb{C}^-$, $\sigma(A + DY_i) \subset \mathbb{C}^+$,

Observe that the antistabilizing solution is computed by applying CR to the equation derived by replacing A, C, D with $-A, -C, -D$, respectively, so that the equation $KT^2 + HT + K^T = 0$ turns into $K^T W^2 + HW + K = 0$ and CR applied to the latter equation substantially coincides with CR applied to the former.

The algorithm is much less expensive than other known methods, in fact it requires only five matrix multiplications and one inversion of a symmetric matrix per step.

In the case in which a stabilizing solution for the ARE exists, algorithm 1 has a quadratic convergence as the following theorem states

Theorem 4. *If continuous-time algebraic Riccati equation (1) admits a stabilizing solution X and an antistabilizing solution Y (i.e., a solution such that $\sigma(A + DY) \subset \mathbb{C}^+$) and CR can by carried out, then $X = X_i + O(\tau^{2^i})$ where $X_i = D^{-1}((T_i + I)(T_i - I)^{-1} - A)$ with $T_i = -\hat{H}_i^{-1} B_{-1}$ and $\mu_n/\mu_{n+1} < \tau < 1$. Here $\mu_i = |\lambda_i + 1|/|\lambda_i - 1|$ and $\lambda_1, \ldots, \lambda_{2n}$ are the eigenvalues of the Hamiltonian matrix arranged so that the sequence $\{\mu_i\}_i$ is nondecreasing.*

Proof. First observe that if Riccati equation has the stabilizing solution X, then the relation

$$\begin{bmatrix} I & O \\ -X & I \end{bmatrix} \begin{bmatrix} A & D \\ C & -A^T \end{bmatrix} \begin{bmatrix} I & O \\ X & I \end{bmatrix} = \begin{bmatrix} A + DX & D \\ O & -A^T - XD \end{bmatrix} \tag{9}$$

is a similarity that transforms the Hamiltonian matrix into a block triangular matrix whose eigenvalues coincide with the ones of $Z = A + DX$ and of $-Z$; the same holds for Y. Then the eigenvalues of the Hamiltonian matrix are

$\sigma(A+DX)\cup\sigma(A+DY)$. The existence of both stabilizing and antistabilizing solutions guarantees that the associated QME equation

$$A_1Z^2+A_0Z+A_{-1} \tag{10}$$

with A_1, A_0 and A_{-1} as in equation (8), has solutions $Z^-=A+DX$ and $Z^+=A+DY$ with eigenvalues on the left and right half complex plane, respectively. In particular, because $\sigma(Z^+)$ and $\sigma(Z^-)$ are disjoint and count n elements each, then eigenvalues of the matrix polynomial associated with equation (10) are exactly $\sigma(Z^+)\cup\sigma(Z^-)$ and then coincide with the ones of the Hamiltonian matrix. Now, $T=-Z^+$ is a solution of $A_1X^2-A_0X+A_{-1}$ with eigenvalues in the left half plane, so for equation (10) the hypotheses of theorem (3) are satisfied so that CR converges and we have $Z=Z^-=Z_i+O(\tau^{2^i})$ where τ is defined in equation (7). It remains to prove that the same relation holds for X. Since $\|X-X_i\|=\|D^{-1}(Z-Z_i)\|\le\|D^{-1}\|\|Z-Z_i\|$, then $X=X_i+O(\tau^{2^i})$ which completes the proof.

Even in the case of discrete-time algebraic Riccati equation, it is possible to design a similar algorithm, under slightly stronger hypotheses. Differently from ARE, here it is not necessary to apply the Cayley transform.

As for the continuous-time case, we put $Z=A-B(R+B^TXB)^{-1}(C+B^TXA)$, because we are interested in the solution with $\sigma(Z)\subset\mathcal{D}^-$. Under the hypotheses that all the matrices are square and that B and $A-B(R+B^TXB)^{-1}(C+B^TXA)$ are invertible, it is easy to prove that Z verifies the quadratic matrix equation

$$A_1Z^2+A_0Z+A_{-1}=0 \tag{11}$$

with $\sigma(Z)\subset\mathcal{D}^-$, and

$$\begin{aligned}
&A_1=CB^{-1}-A^TB^{-T}RB^{-1},\\
&A_0=B^{-T}RB^{-1}+A^TB^{-T}RB^{-1}A+Q-CB^{-1}A-A^TB^{-T}C,\\
&A_{-1}=B^{-T}C-B^{-T}RB^{-1}A.
\end{aligned}$$

The equation (11) is discrete-symmetric and cyclic reduction can be applied directly because we need a solution with eigenvalues inside the unit circle.

The invertibility of $A-B(R+B^TXB)^{-1}(C+B^TXA)$ is related to the existence of the solution of a stabilizing (in discrete sense) solution. The crucial hypothesis is that the matrix B is invertible, and is similar to the analogous hypothesis that D is invertible for the ARE. In fact, in control theory [17] the matrix B of the DARE and the matrix D of the ARE are related to the matrix coefficient of the input function. The condition that the matrix coefficients and the solution be square matrices can be easily relaxed.

We obtain the following algorithm

Algorithm 2. *Cyclic reduction for discrete-time algebraic Riccati equations*

1. Input: A, B, C, Q, R square matrix coefficient of the DARE, B invertible.
2. Put
$A_1 = CB^{-1} - A^T B^{-T} RB^{-1}$, $A_0 = B^{-T} RB^{-1} + A^T B^{-T} RB^{-1} A + Q - CB^{-1} A - A^T B^{-T} C$, $H_0 = A_0, K_0 = A_1, \widehat{H}_0 = H_0$.
3. Repeat
$H_{i+1} = H_i - K_i H_i^{-1} K_i^T - K_i^T H_i^{-1} K_i$, $K_{i+1} = -K_i H_i^{-1} K_i$, $\widehat{H}_{i+1} = \widehat{H}_i - K_i H_i^{-1} K_i^T$
until $\|K_i\| < \varepsilon$, with a given ε and a given matrix norm.
4. Output: $X_{i+1} = B^{-T}(RB^{-1}(A - Z_{i+1}) - C)Z_{i+1}^{-1}$ an approximated stabilizing solution of Riccati equation, where $Z_{i+1} = -\widehat{H}_{i+1}^{-1} A_1$.

4 Singular leading coefficient

If the leading coefficient of the Riccati equation is singular we still may apply our algorithm by performing a suitable preprocessing step. Let us consider the new algebraic Riccati equation

$$\widehat{X}\widehat{D}\widehat{X} + \widehat{A}^T \widehat{X} + \widehat{X}\widehat{A} - \widehat{C} = 0 \tag{12}$$

in which

$$\widehat{D} = \begin{bmatrix} D & I \\ I & 0 \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & -I \\ -I & -I \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} C & -I \\ -I & -2I \end{bmatrix} \tag{13}$$

are $(2n) \times (2n)$ matrices and the leading coefficient $\widehat{D}$ is nonsingular. One can verify that if X is a solution of (1), then the matrix

$$\widehat{X} = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \tag{14}$$

is a solution of (12). Moreover, if X is a stabilizing solution for equation (1), then $\widehat{X}$ is a stabilizing solution for equation (12). In fact

$$\widehat{A} + \widehat{D}\widehat{X} = \begin{bmatrix} A + DX & 0 \\ X - I & -I \end{bmatrix}, \tag{15}$$

and then $\sigma(\widehat{A} + \widehat{D}\widehat{X}) = \sigma(A + DX) \cup \{-1\}$ is a subset of the left half complex plane.

If the stabilizing solution of (12) is unique, then it must coincide with $\widehat{X}$ of (14). In this case, in order to compute the stabilizing solution X of the general continuous-time algebraic Riccati equation (1) it is enough to compute the stabilizing solution $\widehat{X}$ of (12) in which the leading coefficient $\widehat{D}$ in (13) is invertible and then it is sufficient to extract from $\widehat{X}$ the top-left block X which is our solution.

In order to prove the uniqueness of the stabilizing solution of (12) we consider the Hamiltonian matrix of (12)

$$\widehat{H} = \begin{bmatrix} \widehat{A} & \widehat{D} \\ \widehat{C} & -\widehat{A} \end{bmatrix}.$$

Applying the similarity transform (9) to $\widehat{H}$ yields the matrix

$$\begin{bmatrix} \widehat{A} + \widehat{D}\widehat{X} & \widehat{D} \\ 0 & -\widehat{A}^T - \widehat{X}\widehat{D} \end{bmatrix}.$$

From (15) we deduce that $\widehat{H}$ has $2m$ eigenvalues with negative real parts and $2m$ eigenvalues with positive real parts. This implies that the stabilizing solution $\widehat{X}$ of (12) is unique.

For the nonsingularity of $\widehat{D}$ we may reduce (12) to a quadratic matrix equation and apply CR. Convergence of cyclic reduction is still quadratic, in fact theorem 4 still holds under the assumption that equation (12) has an antistabilizing solution besides the stabilizing $\widehat{X}$.

Observe that, since the size of the matrices is doubled, the complexity of CR applied to the new equation (12) is higher. A way to avoid doubling the matrix size is to consider the real Schur form of the symmetric matrix $D = USU^T$, where U is an unitary matrix and S is diagonal of the form $S = \begin{bmatrix} S_p & 0 \\ 0 & 0 \end{bmatrix}$, $S_p = \mathrm{Diag}(d_1, \dots, d_p)$ where p is the rank of the matrix D.

Now, pre-multiplying and post-multiplying (1) by U and U^T, respectively, and replacing D, A, C and X with $\widetilde{D} = S$, $\widetilde{A} = U^T AU$, $\widetilde{C} = U^T CU$ and $\widetilde{X} = U^T XU$, respectively, yields the new equation $\widetilde{X}\widetilde{D}\widetilde{X} + \widetilde{A}^T\widetilde{X} + \widetilde{X}\widetilde{A} - \widetilde{C} = 0$, which has the stabilizing solution $\widetilde{X}$ if the original equation (1) has the stabilizing solution X.

Here, the idea is to enlarge the size of the matrix coefficients at the minimum value which ensures the nonsingularity of the leading coefficient. In order to achieve this goal, let us partition $\widetilde{A}$ and $\widetilde{C}$ into blocks as

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \\ \widetilde{C}_{12}^T & \widetilde{C}_{22} \end{bmatrix},$$

with $\widetilde{A}_{11}, \widetilde{C}_{11} \in \mathbb{R}^{p\times p}$, $\widetilde{A}_{22}, \widetilde{C}_{22} \in \mathbb{R}^{q\times q}$, where $q = n - p$ and the other blocks of suitable dimension.

We consider the new equation

$$\widehat{X}\widehat{D}\widehat{X} + \widehat{A}^T\widehat{X} + \widehat{X}\widehat{A} - \widehat{C} = 0 \tag{16}$$

in which

$$\widehat{D} = \begin{bmatrix} S_p & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_q & 0 \end{bmatrix}, \widehat{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & 0 \\ \widetilde{A}_{21} & \widetilde{A}_{22} & -I_q \\ 0 & -I_q & -I_q \end{bmatrix}, \widehat{C} = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & 0 \\ \widetilde{C}_{21} & \widetilde{C}_{22} & -I_q \\ 0 & -I_q & -2I_q \end{bmatrix}, \tag{17}$$

where I_q is the identity matrix of order q. In this way we obtain an equation whose coefficient are of order $2n-p$ and the matrix $\widehat{D}$ is nonsingular so that algorithm 1 applies.

Observe that, if the matrix D has rank n, i.e., is nonsingular, then equation (16) coincides with the old one. Otherwise, the size is increased by an additive term which is the dimension of the null space of D.

As shown above, if X is a stabilizing solution of equation (1) then $\widehat{X} = \begin{bmatrix} X & 0 \\ 0 & I_q \end{bmatrix}$ is a stabilizing solution of (16). By using similar arguments as before, we may prove that also in this case the enlarged equation has a unique stabilizing solution and that CR can be applied provided that the enlarged equation has an antistabilizing solution besides the stabilizing one.

The above technique can be used for increasing the numerical stability of our algorithm when D is very ill conditioned. In fact, instead of choosing as q the number of null eigenvalues of D, we may set q equal to the number of the small eigenvalues of D which are responsible of the large condition number. In particular, by choosing $q = n$ then $\|\widehat{D}\|_\infty = \|\widehat{D}^{-1}\|_\infty \le 1 + \|D\|_\infty$ so that $\mathrm{cond}_\infty \widehat{D} \le (\|D\|_\infty + 1)^2$, since $\begin{bmatrix} D & I \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ I & -D \end{bmatrix}$.

We are now ready to describe the following algorithm for solving a general continuous-time algebraic Riccati equation

Algorithm 3. *Cyclic reduction algorithm for the solution of continuous-time algebraic Riccati equations*

1. Input: D, A, C with $D^T = D, C^T = C$.
2. Compute a Schur form of $D = USU^T$ in which nonzero eigenvalues are on the left-top.
   Put $\widetilde{D} = S$, $\widetilde{A} = U^T AU$, $\widetilde{C} = U^T CU$.
3. Construct the matrices $\widehat{D}$, $\widehat{A}$, $\widehat{C}$ as in (17).
4. Compute $\widehat{X}$ by means of algorithm 1.
5. Extract $\widetilde{X}$ from $\widehat{X}$, as the leading principal $n \times n$ submatrix.
6. Output: $X = U\widetilde{X}U^T$.

5 Numerical experiments

We have tested our algorithms using Matlab on a personal computer with an Intel Pentium III CPU, at 450 Mhz. We have compared our method with Schur method, Newton method and the matrix sign iteration in terms of both CPU time and residual error. Concerning the precision, we have compared the methods on the basis of the residual $\|\mathcal{R}(X)\| = \|XDX + A^T X + XA - C\|$, or the relative residual $\|\mathcal{R}(X)\|/\|X\|$, where the norm used is the Frobenius norm. A "$*$" in the table means that the method did not converge. We also considered a version of our method where the computation is followed by an iterative refinement made by means of Newton's method in order to get maximal precision.

We considered several test problems among which all the benchmarks of [1]. Here, for space reasons, we report only few results. For more information the reader can directly contact the authors.

Test 1. We generate three matrices $M_1, M_2, M_3 \in \mathbb{R}^{n\times n}$ whose elements are pseudo-random numbers between 0 and 1 and we put $A = M_1,\ C = 0.5(M_2 + M_2^T) + nI,\ D = 0.5(M_3 + M_3^T) + nI$. In this test the method of matrix sign becomes unstable when n is large, because eigenvalues are near to the imaginary axis. Table 1 compares results in terms of CPU time. It appears clear that CR is the fastest method. CR with iterative refinement (CR+IR) has a CPU time which is less than one half of the time required by Newton's method and the precision is the same.

n	CR	Newton	Schur	Sign	CR+IR
20	0.01	0.22	0.38	0.11	0.11
40	0.05	0.71	1.86	0.22	0.27
80	0.27	4.00	12.8	0.94	0.99
160	1.26	36.4	107	$*$	7.2
320	9.06	292	1090	$*$	59

Table 1: Comparison of CPU time (in seconds) of resolution methods in test 1.

Table 2 compares residual obtained by the methods. CR has less precision than Newton method and Schur method, but with iterative refinement it reaches the same precision as Newton method. The matrix sign method fails.

n	CR	Newton	Schur	Sign	CR+IR
20	$6.8\cdot 10^{-14}$	$4.1\cdot 10^{-14}$	$3.5\cdot 10^{-13}$	$1.5\cdot 10^{-13}$	$3.5\cdot 10^{-14}$
40	$2.1\cdot 10^{-13}$	$1.4\cdot 10^{-13}$	$1.6\cdot 10^{-12}$	$6.0\cdot 10^{-13}$	$1.2\cdot 10^{-13}$
80	$1.4\cdot 10^{-12}$	$4.9\cdot 10^{-13}$	$6.6\cdot 10^{-12}$	$2.4\cdot 10^{-12}$	$4.8\cdot 10^{-13}$
160	$4.5\cdot 10^{-12}$	$3.3\cdot 10^{-13}$	$3.4\cdot 10^{-11}$	$***$	$3.4\cdot 10^{-13}$
320	$1.9\cdot 10^{-11}$	$8.7\cdot 10^{-13}$	$2.0\cdot 10^{-10}$	$***$	$8.8\cdot 10^{-13}$

Table 2: Comparison of residuals in test 1.

Test 2. [1] Let consider the ARE $K^TK + A^TX + XA - XBR^{-1}B^TX = 0$, which corresponds to our equation with $D = -BR^{-1}B$ and $C = K^TK$. Let $A = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.02 \end{bmatrix}$, $B = \begin{bmatrix} 0.1 & 0 \\ 0.001 & 0.01 \end{bmatrix}$, $R = \begin{bmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{bmatrix}$, $K = \begin{bmatrix} 10 & 100 \end{bmatrix}$, where R has condition number $\mu(R) = O(1/\varepsilon)$ for $\varepsilon \to 0$. Table

3 reports the norm of the solution, its condition number, the relative residual and the number of steps to reach convergence. In this case CR is robust even for little value of ε, but requires a great number of steps.

parameter	$\|X\|$	$\mu(X)$	$\|\mathcal{R}(X)\|/\|X\|$	steps
$\varepsilon = 1.0$	$9.88 \cdot 10^3$	$4.16 \cdot 10^3$	$3.7 \cdot 10^{-16}$	7
$\varepsilon = 10^{-4}$	$9.40 \cdot 10^3$	$9.90 \cdot 10^4$	$2.9 \cdot 10^{-13}$	7
$\varepsilon = 10^{-8}$	$9.30 \cdot 10^3$	$9.38 \cdot 10^6$	$2.3 \cdot 10^{-8}$	14
$\varepsilon = 10^{-12}$	$9.30 \cdot 10^3$	$9.37 \cdot 10^8$	$3.7 \cdot 10^{-6}$	20
$\varepsilon = 10^{-14}$	$9.29 \cdot 10^3$	$9.68 \cdot 10^9$	$2.2 \cdot 10^{-2}$	23

Table 3: Results of test 2.

It is interesting to study ARE in which the Hamiltonian matrix has eigenvalues in the imaginary axis. In this case a stabilizing solution does not exist, but it could exist a solution having eigenvalues with nonpositive real part. This problem is very hard: Schur method encounters some difficulties, Newton's iteration has linear convergence and matrix sign method can not be applied. The convergence of our algorithm seems to be linear for the initial steps, then at the end seems to turn into quadratic.

Test 3. [14] With the coefficients $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1/4 & 0 \\ 0 & -3/4 \end{bmatrix}$, it turns out that the Hamiltonian matrix has only zero eigenvalue. Riccati equation admits a solution X such that $A + DX$ has no nonpositive eigenvalue. The searched solution is $X = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$. Figure 1 reports the error of Newtons iteration and of CR. The convergence factor for CR seems to be 1/4, and for Newton method 1/2, as observed in [13]. After some steps CR starts the quadratic convergence.

Test 4. Consider a DARE with $A = M_1 + nI$, $B = I$, $C = O$, $Q = 0.5(M_2 + M_2^T)$, $R = 0.5(M_3 + M_3^T)$, where $M_1, M_2, M_3 \in \mathbb{R}^{n\times n}$ have pseudo-random elementss between 0 and 1. Table 4 shows comparison of CPU time. It is clear that our method is much faster than Schur method, and residuals are the same.

n	40	80	160	320
CR	0.11	0.28	1.65	11.6
Schur	0.77	5.71	48.2	416

Table 4: Comparison between CPU time of methods in test 4.

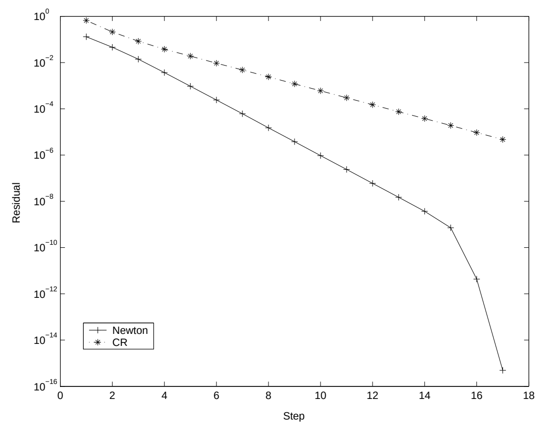

Figure 1: Convergence of iterative methods in test 3

References

[1] P. Benner, A.J. Laub, V. Mehrmann, *A Collection of Benchmark Examples for the Numerical Solution of Algebraic Riccati Equation I: Continuous-Time Case*, Technische Universitt Chemnitz-Zwickau, SFB 393 (Germany). SFB393-Preprint 95-22, 1995

[2] D. Bini, L. Gemignani, B. Meini, *Computations with Infinite Toeplitz Matrices and Polynomial*, Linear Algebra Appl., 343-344 (2002), pp. 21–61.

[3] D. Bini, B. Meini, *On the Solution of a Nonlinear Matrix Equation Arising in Queueing Problems*, Siam J. Matrix Anal. Appl., 17-4 (1996), pp. 906–926.

[4] S. Bittanti, A.J. Laub, et al. *The Riccati Equation*, Springer-Verlag, 1991.

[5] R. Byers, *Solving the Algebraic Riccati Equation with the Matrix Sign Function*, Linear Algebra Appl., 85 (1987), pp. 267–279.

[6] P. Benner, R. Byers, *An Exact Line Search Method for Solving Generalized Continuous-Time Algebraic Riccati Equations*, IEEE Trans. Automat. Control, 43-1 (1998), pp. 101–107.

[7] B.L. Buzbee, G. Golub, C.W. Nielson, *On direct methods for solving Poisson's equation*, SIAM J. Num. Anal., 7 (1970), pp. 627–656.

[8] W. Gander, G. Golub, *Cyclic Reduction – History and Applications*, Scientific computing (Hong Kong, 1997), Springer, Singapore, 1997, pp. 73–85.

[9] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, Academic Press, 1982.

[10] N. Higham, H. Kim, *Numerical Analysis of a Quadratic Matrix Equation*, IMA J. of Numerical Anal., 20 (2000), pp. 499–519.

[11] D. Kleinmann, *On an Iterative Technique for Riccati Equation Computations*, IEEE Trans. Automat. Control, (1968), pp. 114–115.

[12] A.J. Laub, *A Schur Method for Solving Algebraic Riccati Equations*, IEEE Trans. Automat. Control, 24 (1979), pp. 913–921.

[13] P. Lancaster, L. Rodman, *Algebraic Riccati Equations*, Oxford Science Publication, New York, 1995.

[14] V. Larin, *Generalized Matrix Sign Function Algorithm for Algebraic Riccati Equation*, SACTA, 3-2 (2000), pp. 126–137.

[15] G. Latouche, V. Ramaswami, *A logarithmic reduction algorithm for quasi-birth-death processes*, J. Appl. Probab., 30 (1993), pp. 650–674.

[16] G. Latouche and V. Ramaswami, *Introduction to matrix analytic methods in stochastic modeling.* ASA-SIAM Series on Statistics and Applied Probability. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[17] V. Mehrmann, *The Autonomous Linear Quadratic Control Problem*, Lecture Notes in Control and Information Sciences, Vol. 163, Springer Verlag, Berlin. 1991.

[18] J.D. Roberts, *Linear model reduction and solution of the algebraic Riccati equation by use of the sign function*, Internat. J. Contr., 32 (1980), pp. 677–687.